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Stochastic Differential Equations and *a Posteriori* States in Quantum Mechanics

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In recent years a consistent theory describing measurements continuous in time in quantum mechanics has been developed. The result of such a measurement is a "trajectory" for one or more quantities observed with continuity in time. Applications are connected especially with detection theory in quantum optics. In such a theory of continuous measurements one can ask what is the state of the system given that a certain trajectory up to time t has been observed. The response to this question is the notion of *a posteriori* states and a "filtering" equation governing the evolution of such states: this turns out to be a nonlinear stochastic differential equation for density matrices or for pure vectors. The driving noise appearing in such an equation is not an external one, but its probability law is determined by the system itself (it is the probability measure on the trajectory space given by the theory of continuous measurements).

1. INSTRUMENTS AND "*A POSTERIORI*" STATES

In quantum mechanics not only can instantaneous measurements be considered, but so can *measurements continuous in time*. By this we mean the situation in which one or more quantities are followed in their dynamical evolution and probabilities on their "trajectory space" are extracted from quantum mechanics. A consistent theory of measurements continuous in time has been developed (Davies, 1969, 1970, 1971, 1976; Barchielli *et al.*, 1982, 1983, 1985; Lupieri, 1983; Barchielli and Lupieri, 1985*a,b*; Barchielli, 1986*a,b*; Holevo, 1988, 1989) and applications worked out (Srinivas and Davies, 1981, 1982; Holevo, 1982; Barchielli, 1983, 1985, 1987, 1988, 1990, 1991, 1993).

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Now a natural question is: if during a continuous measurement a certain trajectory of the measured observable is registered, what is the state of the system soon after, conditioned upon this information (the *a posteriori* state)? By using ideas from the classical filtering theory for stochastic processes and the formulation of continuous measurements in terms of quantum stochastic differential equations (Barchielli and Lupieri, 1985*a*,*b*; Barchielli, 1986*a*) an Itô stochastic differential equation for the *a posteriori* states has been obtained and solved in some significant cases (Belavkin, 1988, 1989*a*,*b*, 1990*a*,*b*,*c*, 1992; Belavkin and Staszewski, 1989, 1991; Chruściński and Staszewski, 1992; Holevo, 1991; Staszewski and Staszewska, 1992). An alternative derivation is given in Diósi (1988*a*,*b*) and Barchielli and Belavkin (1991).

The aim of this paper is to present the equation for *a posteriori* states in the context of the theory of measurements continuous in time. However, it is interestring to note that some particular forms of the same equation have been introduced in the context of *dynamical reduction theories* (Gisin, 1984, 1986, 1989, 1990; Pearle, 1986; Diósi, 1989; Ghirardi *et al.*, 1989, 1990*a*,*b*, 1991; Nicrosini and Rimini, 1990; Gatarek and Gisin, 1991). Moreover, the same equation, or things related to it, are used for numerical simulations of *master equations* (Carmichael, 1993; Dum *et al.*, 1992*a*,*b*; Gardiner *et al.*, 1992; Mølmer *et al.*, 1993; Gisin and Percival, 1992*a*,*b*), mainly in quantum optics.

Let us start by recalling the important notions of instrument and of *a posteriori* states. The notion of instrument has been introduced in the operational approach to quantum mechanics (Davies and Lewis, 1970). Let a quantum system be described in a separable Hilbert space $\mathscr{H}$ and denote by $\mathscr{B}(\mathscr{H})$ and $\mathscr{T}(\mathscr{H})$ the Banach spaces of the bounded operators on $\mathscr{H}$ and the trace-class operators, respectively. If $\mathscr{A}$ is a linear map on $\mathscr{T}(\mathscr{H})$, its *adjoint* $\mathscr{A}'$ is the linear map on $\mathscr{B}(\mathscr{H})$ defined by $\mathrm{Tr}\{a\mathscr{A}[\varrho]\} = \mathrm{Tr}\{\mathscr{A}'[a]\varrho\}$, $\forall a \in \mathscr{B}(\mathscr{H})$, $\forall \varrho \in \mathscr{T}(\mathscr{H})$. Let (Ω, Σ) be a measurable space (Ω a set and Σ a σ-algebra of subsets of Ω). An *instrument* (Davies and Lewis, 1970; Davies 1976; Ozawa, 1984) (or *operation-valued measure*) $\mathscr{I}$ is a map from Σ into the space of the *linear* bounded operators on $\mathscr{T}(\mathscr{H})$ such that (i) $\mathscr{I}(B)$ is completely positive (Lindblad, 1976) for any $B \in \Sigma$, (ii) $\sum_j \mathscr{I}(B_j)[\varrho] = \mathscr{I}(\bigcup_j B_j)[\varrho]$ for any sequence of pairwise disjoint elements of Σ and any ϱ in $\mathscr{T}(\mathscr{H})$ (convergence in trace norm), and (iii) $\mathrm{Tr}\{\mathscr{I}(\Omega)[\varrho]\} = \mathrm{Tr}\{\varrho\}$, $\forall \varrho \in \mathscr{T}(\mathscr{H})$.

The instrument $\mathscr{I}$ is a measure: (i) is the positivity condition, (ii) is σ-additivity, (iii) is normalization. The instruments represent measurement procedures and their interpretation is as follows. Ω is the set of all possible outcomes of the measurement [(Ω, Σ) is called the value space] and the probability of obtaining the result $\omega \in B$ ($B \in \Sigma$), when before the measurement the system is in a state ϱ [$\varrho \in \mathcal{T}(\mathcal{H})$, $\varrho \geq 0$, $\mathrm{Tr}\{\varrho\} = 1$], is given by $P(B|\varrho) := \mathrm{Tr}\{\mathcal{I}(B)[\varrho]\}$. Let us note that the quantity $\mathcal{I}(B)'[\mathbb{1}]$ turns out to be an effect-valued (Ludwig, 1967) measure, or positive-operator-valued measure, or nonorthogonal resolution of the identity, and it represents an "observable" (Ludwig, 1970, p. 378). Moreover, let us consider a sequence of measurements represented by the instruments $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n$ and performed in the natural order ($\mathcal{I}_2$ after $\mathcal{I}_1$ and so on). We assume any time specification to be included in the definition of the instruments (Heisenberg picture). Then, the joint probability of the sequence of results $\omega_1 \in B_1$, $\omega_2 \in B_2, \ldots, \omega_n \in B_n$, when the *premeasurement state* is ϱ, is given by

$$P(B_1, B_2, \ldots, B_n|\varrho) = \mathrm{Tr}\{\mathcal{I}_n(B_n) \circ \mathcal{I}_{n-1}(B_{n-1}) \circ \cdots \circ \mathcal{I}_1(B_1)[\varrho]\} \quad (1.1)$$

Here $\circ$ means composition of maps. If we consider the conditional probability of the results $\omega_2 \in B_2, \ldots, \omega_n \in B_n$ given the first result $\omega_1 \in B_1$, we can write

$$\begin{aligned} P(B_2, \ldots, B_n|B_1; \varrho) &:= \frac{P(B_1, B_2, \ldots, B_n|\varrho)}{P(B_1|\varrho)} \\ &= P(B_2, \ldots, B_n|\varrho(B_1)) \\ &\equiv \mathrm{Tr}\{\mathcal{I}_n(B_n) \circ \cdots \circ \mathcal{I}_2(B_2)[\varrho(B_1)]\} \end{aligned} \quad (1.2)$$

where we have introduced the statistical operator $\varrho(B_1)$ representing the state after the first measurement, conditioned upon the result $\omega_1 \in B_1$. For a generic instrument $\mathcal{I}$ and set B, the conditioned state $\varrho(B)$ is defined by

$$\varrho(B) = \frac{\mathcal{I}(B)[\varrho]}{\mathrm{Tr}\{\mathcal{I}(B)[\varrho]\}} \equiv \frac{\mathcal{I}(B)[\varrho]}{P(B|\varrho)} \quad (1.3)$$

In particular, by (ii) we obtain $\varrho(\Omega) = \mathcal{I}(\Omega)[\varrho]$. By the definition of instrument, this quantity is linear in ϱ and it is a statistical operator if ϱ is a state. We can call $\varrho(\Omega)$ the *a priori state*: if we know the premeasurement state ϱ and the measurement $\mathcal{I}$, $\varrho(\Omega)$ is the state we can "*a priori*" attribute to our system, before knowing the result of the measurement.

Let us consider now the case when in (1.3) the set B shrinks to an "infinitesimally small" set $d\omega$ around the value ω: the quantity

$$\varrho(\omega) = \frac{\mathcal{I}(d\omega)[\varrho]}{\mathrm{Tr}\{\mathcal{I}(d\omega)[\varrho]\}} \quad (1.4)$$

represents the state conditioned upon the result $\omega \in d\omega$. The quantity $\varrho(\omega)$ is the state one can attribute to those systems for which the result ω has actually been found in the measurement and for this reason we call it the *a posteriori* state.

More precisely, a family of statistical operators $\{\varrho(\omega),\ \omega\in\Omega\}$ is said to be a family of *a posteriori states* (Ozawa, 1985) for an initial state ϱ and an instrument $\mathscr{I}$ with value space $(\Omega,\ \Sigma)$ if (a) the function $\omega\to\varrho(\omega)$ is strongly measurable with respect to the probability measure

$$P(B|\varrho)=\mathrm{Tr}(\mathscr{I}(B)[\varrho]\} \tag{1.5}$$

for the observable associated with the instrument $\mathscr{I}$ and (b) $\forall a\in B(\mathscr{H})$, $\forall B\in\Sigma$,

$$\int_B \mathrm{Tr}\{a\varrho(\omega)\}\ P(d\omega|\varrho)=\mathrm{Tr}\{a\mathscr{I}(B)[\varrho]\} \tag{1.6}$$

For any instrument $\mathscr{I}$ and any premeasurement state ϱ, a family of *a posteriori* states $\varrho(\omega)$ always exists (unique up to equivalence). Let us note that by definition the link between *a priori* and *a posteriori* states is given by

$$\varrho(\Omega)\equiv\mathscr{I}(\Omega)[\varrho]=\int_\Omega \varrho(\omega)\ P(d\omega|\varrho) \tag{1.7}$$

We can say that the *a posteriori* states $\varrho(\omega)$ are a "demixture" of the *a priori* state $\varrho(\Omega)$. Let us stress that (1.6) defines the *a posteriori* states once the instrument $\mathscr{I}$ and the *premeasurement state* ϱ are given. On the contrary, if $\varrho(\omega)$ and $P(d\omega|\varrho)$ are given for any ϱ, (1.6) allows us to reconstruct the instrument $\mathscr{I}$.

A particularly important case is when the instrument $\mathscr{I}$ has a "density" with respect to a numerical measure. Let $Q(d\omega)$ be a numerical measure (possibly a probability measure) on the value space $(\Omega,\ \Sigma)$ and $\mathscr{F}(\omega)$ a family of positive maps on $\mathscr{T}(\mathscr{H})$ such that the instrument $\mathscr{I}$ can be written as

$$\mathscr{I}(B)[\varrho]=\int_B \mathscr{F}(\omega)[\varrho]\ Q(d\omega),\qquad B\in\Sigma,\quad \varrho\in\mathscr{T}(\mathscr{H}) \tag{1.8}$$

Then, by calling ϱ the premeasurement state, we define the nonnormalized *a posteriori* (NNAP) states $\sigma(\omega)$ by

$$\sigma(\omega):=\mathscr{F}(\omega)[\varrho] \tag{1.9}$$

and the probability density $p_\varrho(\omega)$ by

$$p_\varrho(\omega):=\mathrm{Tr}\{\sigma(\omega)\} \tag{1.10}$$

The definitions are such that probabilities and *a posteriori* states are given by

$$P(d\omega|\varrho)=p_\varrho(\omega)\ Q(d\omega),\qquad \varrho(\omega)=\frac{\sigma(\omega)}{p_\varrho(\omega)} \tag{1.11}$$

while the instrument is given by

$$\mathscr{I}(B)[\varrho] = \int_B \sigma(\omega)\, Q(d\omega) = \int_B \varrho(\omega)\, P(d\omega|\varrho) \tag{1.12}$$

From the discussion above and the previous formulas we see that everything can be reconstructed once we have the NNAP states.

Finally, we recall that Ozawa (1984) proved that any instrument can be realized via an "indirect measurement" scheme. Let $\mathscr{H}_0$ be the Hilbert space of an auxiliary system (probe, measuring apparatus, quantum channel, . . .), $\mathrm{Tr}_{\mathscr{H}_0}$ be the partial trace over $\mathscr{H}_0$, ϱ_0 be a state on $\mathscr{H}_0$ (state of the probe), U be a unitary operator on $\mathscr{H} \otimes \mathscr{H}_0$ (dynamics of system and probe), and $P_0(d\omega)$ be a projection-valued measure on $\mathscr{H}_0$ (observable of the probe). Then, any instrument $\mathscr{I}$ on $\mathscr{T}(\mathscr{H})$ can be represented as

$$\mathscr{I}(d\omega)[\varrho] = \mathrm{Tr}_{\mathscr{H}_0}\{U(\varrho \otimes \varrho_0)U^\dagger(\mathbb{1} \otimes P_0(d\omega))\} \tag{1.13}$$

We can say that a measurement on our system can be always realized by letting the system interact with a probe for a certain time and then measuring some observable of the probe. By elimination of the degrees of freedom of the probe, via partial trace, we get the instrument.

2. MEASUREMENTS CONTINUOUS IN TIME

The theory of continuous measurements in the case of quantum point processes (typically, counting of particles) was initiated by Davies (1969, 1970, 1971), while the general formulation of continuous measurements of any kind of observables is due to the Milan group (Barchielli *et al.*, 1982, 1983, 1985; Lupieri, 1983). The idea is to use families of instruments to represent measurements continuous in time. By following Barchielli *et al.* (1982, 1983, 1985) and Holevo (1988, 1989), we formalize continuous measurements by introducing a "trajectory space" Ω [the space of all real-valued—or vector-valued—functions $y(t)$ on $\mathbb{R}$], equipped by a family of σ-algebras of subsets Σ_a^b, $a < b \in \mathbb{R}$ [σ-algebra generated by the increments $y(t) - y(s)$, $a < s \leq t \leq b$]. Then, the measurement in the interval $(a, b]$ is represented by an instrument $\mathscr{I}_a^b$ on the value space (Ω, Σ_a^b). The various instruments have to be compatible in the sense that a measurement in the interval $(a, c]$ must be decomposable in a measurement in the interval $(a, b]$ followed by a measurement in the interval $(b, c]$, $a < b < c$. Precisely, $\forall a, b, c \in \mathbb{R}$, $a < b < c$, $\forall E \in \Sigma_a^b$, $\forall F \in \Sigma_b^c$, we must have

$$\mathscr{I}_b^c(F) \circ \mathscr{I}_a^b(E) = \mathscr{I}_a^c(E \cap F) \tag{2.1}$$

Note that $E \cap F \in \Sigma_a^c$.

A large class of such families of instruments has been constructed by using Fourier transform techniques (Barchielli *et al.*, 1982, 1983, 1985; Lupieri, 1983; Barchielli and Lupieri, 1985*a,b*; Holevo, 1988, 1989). Another way to obtain continuous measurements is by the indirect measurement scheme described by equation (1.13) (Barchielli and Lupieri, 1985*a,b*; Barchielli, 1986*a*); the mathematical techniques one needs in this case are based on "quantum stochastic calculus" (Hudson and Parthasarathy, 1984). This approach is particularly suited for describing detection of photons (direct, heterodyne, homodyne detection) (Barchielli, 1987, 1988, 1990, 1991, 1993). Now the system is a photoemissive source (an atom, an optical cavity, . . .) and the probe is the electromagnetic field in free space; the projection-valued measure in equation (1.13) is now an observable of the field such as number of photons or the field itself. The point is that one can realize these observables through commuting time-dependent self-adjoint operators (let us stress: these operators at different times commute); in other words, one can obtain continuous measurements in the traditional formulation of quantum mechanics.

While the "filtering" equation for *a posteriori* states was originally obtained in Belavkin (1988) by starting from the "quantum stochastic" formulation of Barchielli and Lupieri (1985*a,b*), the approach via *a posteriori* states and NNAP states gives an alternative way of constructing continuous measurements (Barchielli and Belavkin, 1991; Barchielli and Holevo, n.d.).

The idea of this last approach is to give a reference measure Q (in the simplest cases the probability measure of a Poisson or a Wiener process) and then a family of NNAP states $\sigma_t \equiv \sigma_t(\omega; \varrho)$ [cf. equations (1.8)–(1.12)], depending only on the part of the trajectory ω up to time t; ϱ is the initial state of the system. The various quantities must be such that the equation

$$\mathscr{I}_0^t(B)[\varrho] = \int_B \sigma_t(\omega; \varrho)\, Q(d\omega), \qquad B \in \Sigma_0^t, \tag{2.2}$$

[cf. equation (1.12)] truly defines an instrument $\mathscr{I}_0^t$ on (Ω, Σ_0^t). Then, the *a posteriori* states ϱ_t are obtained by normalization

$$\varrho_t = \sigma_t / \mathrm{Tr}\{\sigma_t\} \tag{2.3}$$

Typically, the NNAP states σ_t are given through a *linear stochastic differential equation*; then, the *a posteriori* states ϱ_t satisfy a *nonlinear stochastic differential equation.* While very general cases can be treated by using the general theory of Itô's stochastic calculus, in the next section we present a simple example of diffusive type; a presentation of the counting (or jump) case can be found in Barchielli and Belavkin (1991), where it is also shown how to obtain the diffusive case from the jump case via suitable limits.

3. "*A POSTERIORI*" STATES (DIFFUSIVE PROCESSES)

In this section we develop a theory describing the continuous measurement of d observables; we shall use a heuristic mathematical language, but all statements could be made rigorous. According to the discussion in Section 1, the first step is to introduce a reference measure Q; we take it to be the probability measure for d independent standard Wiener processes. A typical trajectory $W_j(t)$ of these processes is a nondifferentiable continuous function such that

$$W_j(0) = 0, \qquad dW_j(t)\, dW_i(t) = \delta_{ji}\, dt, \qquad dt\, dW_j(t) = 0 \tag{3.1}$$

The increments are always intended to point into the future, $dW_j(t) := W_j(t + dt) - W_j(t)$ (Itô's definition of stochastic integrals).

The second step is to give an equation for the NNAP states σ_t. We first write the equation and then we discuss its properties. Let us consider the Itô stochastic differential equation

$$d\sigma_t = \mathscr{L}[\sigma_t]\, dt + \sum_{j=1}^{d} (L_j\sigma_t + \sigma_t L_j^*)\, dW_j(t) \tag{3.2}$$

where

$$\mathscr{L}[\sigma] := \mathscr{L}_0[\sigma] + \sum_{j=1}^{d} \left(L_k\sigma L_k^* - \frac{1}{2}\{L_k^* L_k, \sigma\}\right) \tag{3.3}$$

$$\mathscr{L}_0[\sigma] := -i[H, \sigma] + \sum_r \left(R_r\sigma R_r^* - \frac{1}{2}\{R_r^* R_r, \sigma\}\right) \tag{3.4}$$

$L_k, H, R_r \in B(\mathscr{H})$, $H = H^\dagger$, $\{a, b\} := ab + ba$.

Let us consider as initial condition of equation (3.2) $\sigma_0 = \varrho$, where ϱ is the initial state of the system. A first property of equation (3.2) is that its solution depends only on the past (it is *adapted*) and therefore σ_t and $dW_j(t)$ are independent (the Wiener process has independent increments). A second property is that indeed our equation transforms positive trace-class operators into positive trace-class operators.

Let us denote by $\mathbb{E}_Q$ the mathematical expectation with respect to the probability measure Q: $\mathbb{E}_Q[f] = \int_\Omega f(\omega)\, Q(d\omega)$; we recall that $\mathbb{E}_Q[W_j(t)] = 0$. According to the first part of equation (1.12) with $B = \Omega$, the quantity

$$\Theta_t := \mathbb{E}_Q[\sigma_t] \tag{3.5}$$

is the *a priori* state at time t. By taking the expectation of both sides of equation (3.2), it turns out that the *a priori* states satisfy the *master equation*

$$\frac{d}{dt}\Theta_t = \mathscr{L}[\Theta_t] \tag{3.6}$$

with *Liouvillian* (3.3). In the mean, the system undergoes a dissipative dynamics.

Let us now take the trace of both sides of equation (3.2). By noting that $\mathrm{Tr}\{\mathscr{L}[\sigma]\} = 0$, we get

$$dp_t = p_t \sum_{j=1}^{d} m_j(t)\, dW_j(t), \qquad p_t := \mathrm{Tr}\{\sigma_t\}, \qquad m_j(t) := 2\,\mathrm{Re}\,\mathrm{Tr}\{L_j\varrho_t\} \tag{3.7}$$

ϱ_t is the *a posteriori* state defined by equation (2.3). By taking the expectation of equation (3.7), we see that $\mathbb{E}_Q[p_t]$ is a constant. Because p_t is positive and $\mathbb{E}_Q[p_0] = 1$, we have that p_t is a probability density with respect to Q. According to equations (1.10) and (1.11), the probability measure of the measuring process is

$$P_t(d\omega|\varrho) := p_t(\omega)\, Q(d\omega) \tag{3.8}$$

P_t is the measure on the σ-algebra of trajectories up to time t and $P := \lim_{t\to+\infty} P_t$ gives the probability measure on the space of all trajectories.

Let us now consider equation (3.2) with initial condition $\sigma_s = \varrho$ at time s and call $\Lambda_s^t[\varrho]$ its solution. By the first of equations (1.12) we define for $E \in \Sigma_s^t$, $s < t$,

$$\mathscr{I}_s^t(E)[\varrho] = \int_E \Lambda_s^t(\omega)[\varrho]\, Q(d\omega) \tag{3.9}$$

It turns out that indeed $\{\mathscr{I}_s^t\}$ is a family of instruments satisfying equation (2.1). Note that $\mathscr{I}_0^t(E)[\varrho] = \int_E \sigma_t(\omega)\, Q(d\omega)$.

Finally, let us find an equation for the *a posteriori* states (2.3). By Itô's formula, from equation (3.7) we obtain

$$dp_t^{-1} = p_t^{-1} \sum_{j=1}^{d} [m_j(t)^2\, dt - m_j(t)\, dW_j(t)] \tag{3.10}$$

and from equations (3.2) and (3.10) we obtain

$$d\varrho_t = \mathscr{L}[\varrho_t]\, dt + \sum_{j=1}^{d} [L_j\varrho_t + \varrho_t L_j^* - m_j(t)\varrho_t][dW_j(t) - m_j(t)\, dt] \tag{3.11}$$

This is a nonlinear equation because the $m_j(t)$ depend on ϱ_t itself. An important point is that, according to equation (1.5) and the second of equations (1.12), the *a posteriori* states are distributed with probability law P.

By using Itô's table (3.1), we can compute the differential of $p_t W_j(t)$; we get

$$d(p_t W_j(t)) = p_t \left\{ \sum_{i=1}^{d} [\delta_{ji} + W_j(t) m_i(t)]\, dW_i(t) + m_j(t)\, dt \right\} \qquad (3.12)$$

By the fact that the various processes are adapted, we have $\mathbb{E}_P[W_j(t)] = \mathbb{E}_Q[W_j(t) p_t]$ and

$$\frac{d}{dt} \mathbb{E}_P[W_j(t)] = \mathbb{E}_P[m_j(t)] \qquad (3.13)$$

First, this equation and equation (3.11) show that $\mathbb{E}_P[\varrho_t]$ satisfies the master equation (3.6) and, by taking the same initial condition, we get

$$\Theta_t = \mathbb{E}_P[\varrho_t] \qquad (3.14)$$

which is the analog of equation (1.7). Moreover, by equation (3.13) and the theory of the Girsanov transform (Gatarek and Gisin, 1991), we have that *under the probability law* $P(\cdot\,|\varrho)$ the processes

$$B_j(t) := \int_{(0,t]} [dW_j(s) - m_j(s)\, ds] \qquad (3.15)$$

are *d independent standard Wiener processes.*

By equations (3.7)–(3.9) we have that the results of the measurements $\{\mathscr{F}_s^t\}$ are distributed with law P. Due to the arbitrariness of the initial condition $W_j(0) = 0$, the output of the measurement can be identified with the increments of the processes $W_j(t)$ under the law P, or better, the measured observables are the time derivatives $dW_j(t)/dt$ (to be understood in the sense of generalized stochastic processes) (Barchielli *et al.*, 1982, 1983, 1985). By putting together equations (3.6), (3.13), and (3.14) and the last of equation (3.7) we get

$$\frac{d}{dt} \mathbb{E}_P[W_j(t)] = \mathrm{Tr}\{(L_j + L_j^*) e^{\mathscr{L}t}[\varrho]\} \qquad (3.16)$$

We can interpret this equation by saying that we are making a continuous imprecise measurement of the observables represented by the self-adjoint operators $L_j + L_j^*$, which are in general noncommuting. Moreover, we can read equation (3.15) as "the observed quantities $\dot{W}_j(t)$ are decomposed into the sum of the white noises $\dot{B}_j(t)$ plus the processes $m_j(t)$."

Equations (3.2) and (3.11) have an equivalent pure-state version. In the case that $\mathscr{L}_0$ of (3.4) is not of purely Hamiltonian form we need further independent standard Wiener processes $W_r(t)$. The pure-state version of equation (3.2) is

$$d\psi_t = \left[\sum_{j=1}^{d} L_j \, dW_j(t) + \sum_r R_r \, dW_r(t) - \left(iH + \frac{1}{2} \sum_{j=1}^{d} L_j^* L_j + \frac{1}{2} \sum_r R_r^* R_r \right) dt \right] \psi_t \tag{3.17}$$

Indeed, by setting $\tilde{\sigma}_t = |\psi_t\rangle\langle\psi_t|$, from Itô's table (3.1) and equation (3.17) we obtain

$$d\tilde{\sigma}_t = \mathscr{L}[\tilde{\sigma}_t] \, dt + \sum_{j=1}^{d} (L_j \tilde{\sigma}_t + \tilde{\sigma}_t L_j^*) \, dW_j(t) + \sum_r (R_r \tilde{\sigma}_t + \tilde{\sigma}_t R_r^*) \, dW_r(t) \tag{3.18}$$

By taking the stochastic mean over the auxiliary Wiener processes $W_r(t)$ we end up with equation (3.2).

By setting $\hat{\psi}_t = \psi_t / \|\psi_t\|$, we obtain

$$\begin{aligned} d\hat{\psi}_t = \Big\{ & -iH \, dt + \sum_{j=1}^{d} [L_j - \tfrac{1}{2}\hat{m}_j(t)] \, dY_j(t) \\ & + \sum_r [R_r - \tfrac{1}{2}\hat{n}_r(t)] \, dZ_r(t) \\ & - \frac{1}{2} \sum_{j=1}^{d} [L_j^* L_j - \hat{m}_j(t) L_j + \tfrac{1}{4}\hat{m}_j(t)^2] \, dt \\ & - \frac{1}{2} \sum_r [R_r^* R_r - \hat{n}_r(t) R_r + \tfrac{1}{4}\hat{n}_r(t)^2] \, dt \Big\} \hat{\psi}_t \end{aligned} \tag{3.19}$$

where

$$\hat{m}_j(t) = 2 \, \mathrm{Re}\langle \hat{\psi}_t | L_j \hat{\psi}_t \rangle, \qquad \hat{n}_r(t) = 2 \, \mathrm{Re}\langle \hat{\psi}_t | R_r \hat{\psi}_t \rangle \tag{3.20}$$

$$dY_j(t) = dW_j(t) - \hat{m}_j(t) \, dt, \qquad dZ_r(t) = dW_r(t) - \hat{n}_r(t) \, dt \tag{3.21}$$

Equation (3.19) is the pure-state version of equation (3.11). Indeed, by setting $\tilde{\varrho}_t = |\hat{\psi}_t\rangle\langle\hat{\psi}_t|$, we obtain

$$\begin{aligned} d\tilde{\varrho}_t = & \mathscr{L}[\tilde{\varrho}_t] \, dt + \sum_{j=1}^{d} [L_j \tilde{\varrho}_t + \tilde{\varrho}_t L_j^* - \hat{m}_j(t) \tilde{\varrho}_t] \, dY_j(t) \\ & + \sum_r [R_r \tilde{\varrho}_t + \tilde{\varrho}_t R_r^* - \hat{n}_r(t) \tilde{\varrho}_t] \, dZ_r(t) \end{aligned} \tag{3.22}$$

By taking the conditional expectation with respect to the σ-algebra generated by the processes $Y_j(t)$, we have $dZ_r(t) \to 0$, $\hat{m}_j(t) \to m_j(t)$ and we obtain equation (3.11).

It is interesting to note that the first example of the type of equation (3.19) (diffusive case) was introduced by Gisin (1984) in the context of dynamical reduction theories, while examples of the other equations (linear and nonlinear versions), including the case of counting processes (jump case), were introduced by Belavkin (1988, 1989*a*) in the context of continuous measurement theory; for the general case, see Belavkin (1990*c*) and Barchielli and Holevo (n.d.). Moreover, equations like (3.19) are now used for numerical simulations of the corresponding master equation (3.6) (Gisin and Percival, 1992*a*,*b*). Somewhat similar simulations are used in the jump case (Carmichael, 1993; Dum *et al.*, 1992*a*,*b*; Gardiner *et al.*, 1992; Mølmer *et al.*, 1993), but without explicit reference to a stochastic equation.

REFERENCES

Barchielli, A. (1983). *Nuovo Cimento*, **74B**, 113–138.
Barchielli, A. (1985). *Physical Review D*, **32**, 347–367.
Barchielli, A. (1986*a*). *Physical Review A*, **34**, 1642–1649.
Barchielli, A. (1986*b*). In *Fundamental Aspects of Quantum Theory*, V. Gorini and A. Frigerio, eds., Plenum Press, New York, pp. 65–73.
Barchielli, A. (1987). *Journal of Physics A: Mathematical and General*, **20**, 6341–6355.
Barchielli, A. (1988). In *Quantum Probability and Applications III*, L. Accardi and W. von Waldenfels, eds., Springer, Berlin, pp. 37–51.
Barchielli, A. (1990). *Quantum Optics*, **2**, 423–441.
Barchielli, A. (1991). In *Quantum Aspects of Optical Communications*, C. Bendjaballah, O. Hirota, and S. Reynaud, eds., Springer, Berlin, pp. 179–189.
Barchielli, A. (1993). In *Classical and Quantum Systems—Foundations and Symmetries*, H. D. Doebner, W. Scherer, and F. Schroeck, Jr., eds., World Scientific, Singapore, p. 488.
Barchielli, A., and Belavkin, V. P. (1991). *Journal of Physics A: Mathematical and General*, **24**, 1495–1514.
Barchielli, A., and Holevo, A. S. (n.d.). Constructing quantum processes via classical stochastic calculus, preprint.
Barchielli, A., and Lupieri, G. (1985*a*). *Journal of Mathematical Physics* **26**, 2222–2230.
Barchielli, A., and Lupieri, G. (1985*b*). In *Quantum Probability and Applications II*, L. Accardi and W. von Waldenfels, eds., Springer, Berlin, pp. 57–66.
Barchielli, A., Lanz L., and Prosperi, G. M. (1982). *Nuovo Cimento*, **72B**, 79–121.
Barchielli, A., Lanz, L., and Prosperi, G. M. (1983). *Foundations of Physics*, **13**, 779–812.
Barchielli, A., Lanz, L., and Prosperi, G. M. (1985). In *Chaotic Behavior in Quantum Systems*, G. Casati, ed., Plenum Press, New York, pp. 321–344.
Belavkin, V. P. (1988). In *Modelling and Control of Systems*, A. Blaquière, ed., Springer, Berlin, pp. 245–265.
Belavkin, V. P. (1989*a*). *Physics Letters A*, **140**, 355–358.
Belavkin, V. P. (1989*b*). *Journal of Physics A: Mathematical and General*, **22**, L1109–L1114.
Belavkin, V. P. (1990*a*). *Journal of Mathematical Physics*, **31**, 2930–2934.
Belavkin, V. P. (1990*b*). *Letters in Mathematical Physics*, **20**, 85–89.

Belavkin, V. P. (1990*c*). In *Probability Theory and Mathematical Statistics*, B. Grigelionis *et al.*, eds., Mokslas, Vilnius, and VSP, Utrecht, Volume 1, pp. 91–109.
Belavkin, V. P. (1992). *Communications in Mathematical Physics*, **146**, 611–635.
Belavkin, V. P., and Staszewski, P. (1989). *Physics Letters A*, **140**, 359–362.
Belavkin, V. P., and Staszewski, P. (1991). *Reports on Mathematical Physics*, **29**, 213–225.
Carmichael, H. (1993). *An Open Systems Approach to Quantum Optics*, Springer, Berlin.
Chruściński, D., and Staszewski, P. (1992). *Physica Scripta*, **45**, 193–199.
Davies, E. B. (1969). *Communications in Mathematical Physics*, **15**, 277–304.
Davies, E. B. (1970). *Communications in Mathematical Physics*, **19**, 83–105.
Davies, E. B. (1971). *Communications in Mathematical Physics*, **22**, 51–70.
Davies, E. B. (1976). *Quantum Theory of Open Systems*, Academic Press, London.
Davies, E. B., and Lewis, J. T. (1970). *Communications in Mathematical Physics*, **17**, 239–260.
Diósi, L. (1988*a*). *Physics Letters A*, **129**, 419–423.
Diósi, L. (1988*b*). *Physics Letters A*, **132**, 233–236.
Diósi, L. (1989). *Physical Review A*, **40**, 1165–1174.
Dum, R., Zoller, P., and Ritsch, H. (1992*a*). *Physical Review A*, **45**, 4879–4887.
Dum, R., Parkins, A. S., Zoller, P., and Gardiner, C. W. (1992*b*). *Physical Review A*, **46**, 4382–4396.
Gardiner, C. W., Parkings, A. S., and Zoller, P. (1992). *Physical Review A*, **46**, 4363–4381.
Gatarek, D., and Gisin, N. (1991). *Journal of Mathematical Physics*, **32**, 2152–2157.
Ghirardi, G. C., Pearle, P., and Rimini, A. (1989). In *Proceedings 3rd International Symposium on the Foundations of Quantum Mechanics*, Tokyo, pp. 181–189.
Ghirardi, G. C., Pearle, P., and Rimini, A. (1990*a*). *Physical Review A*, **42**, 78–89.
Ghirardi, G. C., Pearle, P., and Rimini, A. (1990*b*). In *Sixty-Two Years of Uncertainty*, A. I. Miller, ed., Plenum Press, New York, pp. 167–191.
Ghirardi, G. C., Grassi, R., and Pearle, P. (1991). In *Symposium on the Foundations of Modern Physics 1990*, P. Lahti and P. Mittelstaedt, eds., World Scientific, Singapore, pp. 109–123.
Gisin, N. (1984). *Physics Review Letters*, **52**, 1657–1660.
Gisin, N. (1986). *Journal of Physics A: Mathematical and General*, **19**, 205–210.
Gisin, N. (1989). *Helvetica Physica Acta*, **62**, 363–371.
Gisin, N. (1990). *Helvetica Physica Acta*, **63**, 929–939.
Gisin, N., and Percival, I. C. (1992*a*). *Physics Letters A*, **167**, 315–318.
Gisin, N., and Percival, I. C. (1992*b*). *Journal of Physics A: Mathematical and General*, **25**, 5677–5691.
Holevo, A. S. (1982). *Soviet Mathematics*, **26**, 1–20 [*Izvestiya Vysshikh Uchebnykh Zavedenii. Seriya Matematika*, **26**, 3–19].
Holevo, A. S. (1988). In *Quantum Probability and Applications III*, L. Accardi and W. von Waldenfels, eds., Springer, Berlin, pp. 128–148.
Holevo, A. S. (1989). In *Quantum Probability and Applications IV*, L. Accardi and W. von Waldenfels, eds., Springer, Berlin, pp. 229–255.
Holevo, A. S. (1991). In *Quantum Aspects of Optical Communications*, C. Bendjaballah *et al.*, eds., Springer, Berlin, pp. 127–137.
Hudson, R. L., and Parthasarathy, K. R. (1984). *Communications in Mathematical Physics*, **93**, 301–323.
Lindblad, G. (1976). *Communications in Mathematical Physics*, **48**, 119–130.
Ludwig, G. (1967). *Communications in Mathematical Physics*, **4**, 331–348.

Ludwig, G. (1970). *Deutung des Begriffs "physikalische" Theorie und axiomatische Grundlegung der Hilbertraumstruktur der Quantenmechanik durch Hauptsätze des Messes*, Springer, Berlin.

Lupieri, G. (1983). *Journal of Mathematical Physics*, **24**, 2329–2339.

Mølmer, K., Castin, Y., and Dalibard, J. (1993). *Journal of the Optical Society of America*, **B10**, 524–538.

Nicrosini, O., and Rimini, A. (1990). *Foundations of Physics*, **20**, 1317–1327.

Ozawa, M. (1984). *Journal of Mathematical Physics*, **25**, 79–87.

Ozawa, M. (1985). *Publications RIMS, Kyoto University*, **21**, 279–295.

Pearle, P. (1986). *Physical Review D*, **33**, 2240

Srinivas, M. D. and Davies, E. B. (1981). *Optica Acta*, **28**, 981–996.

Srinivas, M. D., and Davies, E. B. (1982). *Optica Acta*, **29**, 235–238.

Staszewski, P., and Staszewska, G. (1992). *Open Systems Information Dynamics*, **1**, 103–114.